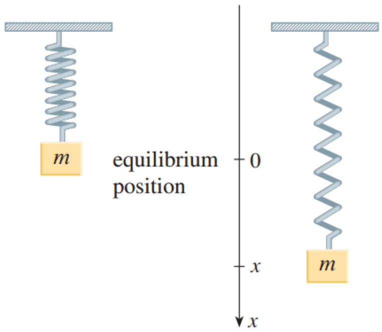
Ordinary Differential Equation (ODE): Initial-Value Problems

Objectives

- Knowing how to implement the Euler's and Runge-Kutta (RK) methods for the first-order and the second-order ODEs.

Most of physical problems are described through the differential equations. For example, we can describe the movement of a spring-mass system from Hook's Law:



$$F = kx \qquad (1)$$

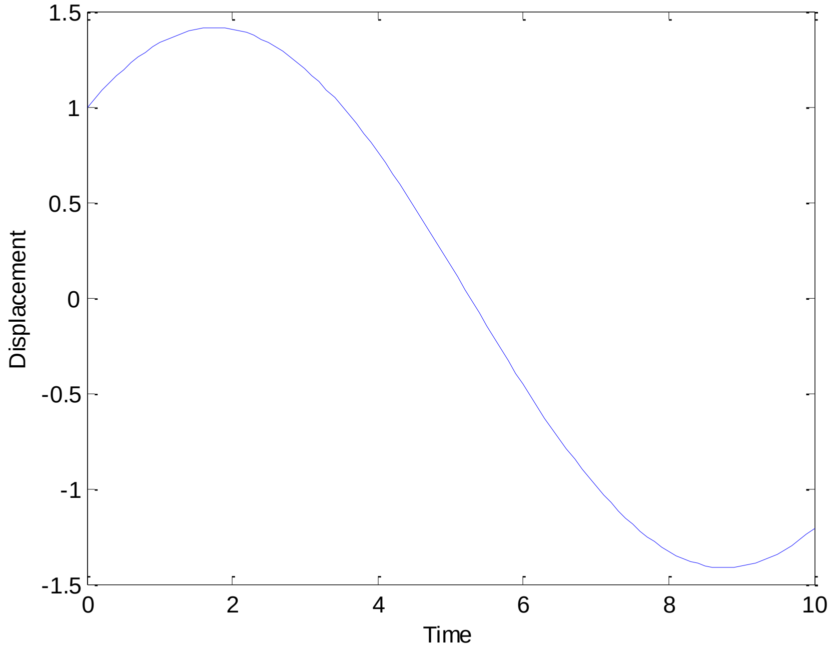
Where F = force, k = a spring constant, and x = the displacement of spring. From Newton's Second Law, we have

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad (2)$$

Using an analytical method, the general solution of Eq. (2) is

$$x(t) = c_1 \cos\sqrt{\frac{k}{m}}t + c_2 \sin\sqrt{\frac{k}{m}}t \qquad (3)$$

The example of the spring movement obtained from Eq. (3) is shown in the following figure.



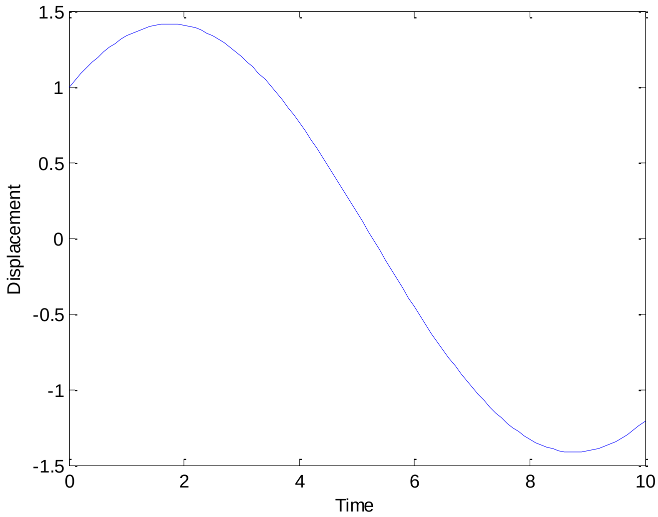
Euler's Method

This section is devoted to solving the first-order ordinary differential equations of the form

$$\frac{dy}{dt} = f(t, y) \tag{4}$$

With initial condition $y(t_0) = y_0$.

This method estimates y_{i+1} from y_i by performing the linear extrapolation as follows:



The slop of the estimated line is

$$slope = \frac{dy}{dt} = \frac{y_{i+1} - y_i}{t_{i+1} - t_i} = f(t_i, y_i) \tag{5}$$

To estimate y_{i+1} from Eq. 95), we have

$$y_{i+1} = y_i + f(t_i, y_i)(t_{i+1} - t_i) \tag{6}$$

Let $t_{i+1} - t_i$ = h which is a step size for estimation.

Thus, Eq. (6) can be written as

$$y_{i+1} = y_i + f(t_i, y_i)h \tag{7}$$

The estimation in Eq. (7) can be expressed in the general form of approximation as

$$y_{i+1} = y_i + \phi h \tag{8}$$

Where ϕ is an incremental function.

According to the similarity between the approximation using Euler's Method and the general form of approximation, the incremental function, ϕ, for Euler's Method is

$$\phi = f(t_i, y_i) \tag{9}$$

Example 1 Solve the following ODE using Euler's Method

$$\frac{dy}{dx} = 3y + x, \qquad y(0) = 1, \quad h = 1$$

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Runge-Kutta (RK) Method

From the the general form of approximation,

$$y_{i+1} = y_i + \phi h \tag{8}$$

For Runge-Kutta (RK) Method, the incremental function can be written in general form as

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n \tag{9}$$

Where the a's are constant and the k's are

$$k_1 = f(t_i, y_i) \tag{9a}$$

$$k_2 = f(t_i + p_1 h, \quad y_i + q_{11} k_1 h) \tag{9b}$$

$$\begin{gathered} k_3 = f(t_i + p_2 h, \quad y_i + q_{21} k_1 h + q_{22} k_2 h) \\ \vdots \end{gathered} \tag{9c}$$

$$k_n = f(t_i + p_{n-1} h, \quad y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h) \tag{9d}$$

Where the p's and q's are constants.

The most popular RK method are fourth order. **The general form for the fourth-order RK method** is given by

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \tag{10}$$

Where

$$k_1 = f(t_i, y_i) \tag{11}$$

$$k_2 = f(t_i + \frac{1}{2}h, \quad y_i + \frac{1}{2}k_1 h) \tag{12}$$

$$k_3 = f(t_i + \frac{1}{2}h, \quad y_i + \frac{1}{2}k_2 h) \tag{13}$$

$$k_4 = f(t_i + h, \quad y_i + k_3 h) \tag{14}$$

Example 2 Use the fourth RK method to solve $y' = 4e^{0.8t} - 0.5y$ from t = 0 to 1 using a step size of 1 with y(0) = 2.

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<u>The second-order ODEs</u>

To solve a second-order ODE, we need **to transform the second-order ODE to a pair of the first-order ODEs** by prescribing the intermediate variable and substituting to the second-order ODE. **After that, we can solve the first-order ODEs simultaneously by using Euler's method or RK method.**

Example 3 Use Euler's method to solve $y'' + 2y' - 3y = 0$ from t = 0 to 2 using a step size of 1 with y(0) = 1 and y'(0)=1.

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Exercise

1. Solve the following initial value problem over the interval from t=0 to 2.

$$\frac{dy}{dt} = yt^3 - 1.5y$$

   a) Using analytical method.

   b) Using Euler's method with h = 0.5 and 0.25.

   c) Using the fourth-order RK method with h = 0.5.

2. The van der Pol equation is a model of an electronic circuit that arose back in the days of vacuum tubes:

$$\frac{d^2y}{dt^2} - (1-y^2)\frac{dy}{dt} + y = 0$$

   Given the initial conditions, y(0) =y'(0)=1, solve this equation from t = 0 to 10 using Euler's method with a step size of **(a)** 0.2 and **(b)** 0.1.

3. Solve the following pair of OOEs over the interval from t =0 to 0.4 using a step size of 0.1. The initial conditions are y(0) = 2 and z(0) = 4. Obtain your solution with (a) Euler's method and (b) the fourth-order RK method.

$$\frac{dy}{dt} = -2y + 5ze^{-1}$$

$$\frac{dz}{dt} = -\frac{yz^2}{2}$$